Radiation reaction and renormalization in classical electrodynamics of a point particle in any dimension

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The effective equations of motion for a point charged particle taking into account the radiation reaction are considered in various space-time dimensions. The divergences stemming from the pointness of the particle are studied and an effective renormalization procedure is proposed encompassing uniformly the cases of all even dimensions. It is shown that in any dimension the classical electrodynamics is a renormalizable theory if not multiplicatively beyond $d=4$. For the cases of three and six dimensions the covariant analogues of the Lorentz-Dirac equation are explicitly derived.

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I. INTRODUCTION

The problem of accounting for the radiation back reaction to the relativistic motion of a point charge has been the subject of intensive studies since Dirac's seminal paper [1]. The equation of motion proposed by Dirac coincides in the nonrelativistic limit with the zero-size limit of the Lorentz model for an electron [2] and for this reason it is often referred to as the Lorentz-Dirac (LD) equation. For a modern review and further references, see [3–7]. Apart from pure theoretical interest, the LD equation finds applications in the physics of accelerators and astrophysics [8].

In this paper we formulate the general framework for deriving the LD equation in arbitrary dimension space-times. The main problem one comes up against when trying to consistently derive the effective equation of motion for a point charge is the inevitable infinities arising due to the "pointness" of the particle. The elimination procedure for these "classical" divergencies relies on the same renormalization philosophy that is used in the quantum field theory and in this case one may put it in a rigorous mathematical framework. The point is that, as far as classical particle motion is concerned, one has to regularize the equations of motion, which are linear in the field. This may require one to remove the infinities only from a single Green's function (which determines the retarded Liénard and Wiechert potentials); no products of such functions are needed. We show that in this linear situation the regularization problem is resolved by the standard tools of classical functional analysis, without invoking the more powerful but less rigorous machinery of quantum renormalization theory. As a result we establish the general structure of the Lorentz-Dirac equations for any even d as well as the counterterms needed to compensate all the divergences. In the particular case of $d=6$ the results of our analysis are in good agreement with those of [9] where the six-dimensional LD equation was obtained on the basis of energy conservation and reparametrization invariance arguments.

When this work had been completed, we learned about the paper [10] where the question of the radiation reaction in various dimensions is discussed. In this paper, the distinctions are noted between even and odd dimensions and the radiation reaction force is derived in $d=3$. It was also argued that $d=3,4$ are the only dimensions where all the divergences are removed by the mass renormalization, which checks well with our analysis and the previous studies of [9]. It should be noted, however, that the form of the 3D integro-differential LD equation proposed in [10] differs in appearance from that derived in the present paper.

The paper is organized as follows. In Sec. II we set the notation and discuss the difference between the retarded Green's functions in odd and even dimensions. The regularization procedure for the singular linear functionals relevant to our problem is detailed in Sec. III. In Sec. IV, this technique is applied to deriving the LD equation in various dimensions. Starting with the case of even dimensions we develop a convenient regularization scheme for deriving the four-dimensional LD equation and yet allowing generation of its higher dimensional analogues by mere expansion in a regularization parameter. The case of odd dimensions seems to be less interesting as it leads to the *nonlocal* (integro-differential) LD equation, so after a discussion of the general structure of the LD force we restrict ourselves by considering the simple example of a $2+1$ particle. The main result of this section is that in any dimension the infinities coming from the particle's self-action can be compensated by a finite number of counterterms added to the original action functional, which means the renormalizability of classical electrodynamics. In the concluding section we summarize the results and outline the prospects for further investigations.

II. EQUATIONS OF MOTION AND GREEN'S FUNCTION

In this section we recall some basic formulas concerning the one-particle problem of classical electrodynamics formulated in an arbitrary dimensional space-time. The detailed treatment of the subject can be found, for example, in [11].

So, let $\mathbb{R}^{d-1,1}$ be d-dimensional Minkowski space with coordinates x^μ, $\mu=0,\dots,d-1$, and signature

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$(+-\cdots-)$. Consider a scalar point particle of mass m and charge e coupled to the electromagnetic field. The dynamics of the whole system (field)+(particle) is governed by the action functional

$$S=-\frac{N_d}{4}\int d^dxF_{\mu\nu}F^{\mu\nu}+e\int d\tau A_\mu\dot{x}^\mu-m\int d\tau\sqrt{\dot{x}^2},$$
$$N_d=\frac{\pi^{(1-d)/2}}{2}\Gamma^{-1}\left(\frac{d-1}{2}\right), \tag{1}$$

where $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$ is the strength tensor of the electromagnetic field A_μ and the overdot means the derivative with respect to the particle's proper time τ.[1] Hereafter we use the natural system of units $(c=1,\hbar=1)$ so that $[m]=-[x]=1$ and $[e]=1-[A]=(4-d)/2$.

The variation of the action (1) results in a coupled system of the Maxwell and Lorentz equations describing both the motion of a charged point particle in response to the electromagnetic field and the propagation of the electromagnetic field produced by the moving charge. In the Lorentz gauge $\partial^\mu A_\mu=0$ the equations take form

$$\Box A_\mu(x)=-N_d^{-1}j_\mu(x),\quad D^2x^\mu=\mathcal{F}^\mu \tag{2}$$

where the right-hand sides of the equations are given by the electric current of the point particle moving along the world line $x^\mu(\tau)$ and the Lorentz force

$$j^\mu(x)=e\int d\tau\delta(x-x(\tau))\dot{x}^\mu(\tau)$$
$$\mathcal{F}_\mu=\frac{e}{m}F_{\mu\nu}(x)Dx^\nu. \tag{3}$$

Hereafter we use an invariant derivative

$$D=\frac{1}{\sqrt{\dot{x}^2}}\frac{d}{d\tau} \tag{4}$$

whose repeat action $D^nx^\mu(\tau)$ on a trajectory remains intact under reparametrization: $\tau\rightarrow\tau'(\tau)$.

It is well known that the character of propagation of electromagnetic waves depends strongly on the space-time dimension d, and especially on its parity. Mathematically, this manifests in quite different expressions for the *retarded* Green's function $G=\Box^{-1}$ associated with the D'Alembert operator:

$$G(x)=\begin{cases}\dfrac{1}{2}\pi^{(2-d)/2}\theta(x_0)\,\delta^{(d/2-2)}(x_0^2-\mathbf{x}^2) & \text{for}\ \ d=4,6,8,\ldots,\\[2ex] \dfrac{(-1)^{(d-3)/2}}{2}\pi^{-d/2}\Gamma\left(\dfrac{d-2}{2}\right)\theta(x_0-|\mathbf{x}|)(x_0^2-\mathbf{x}^2)^{(2-d)/2} & \text{for}\ \ d=3,5,7,\ldots,\end{cases} \tag{5}$$

where $\mathbf{x}=(x^1,\ldots,x^{d-1})$. In an even-dimensional space-time the Green's function is localized on a forward light cone with the vertex at the origin, while for the case of odd dimensions its support extends to the interior of the light cone. These distinctions have a crucial physical consequence, which can be illustrated by the following gedanken experiment. Suppose a source of light is turned on at an initial time t and is then turned off at a time t'. If the number of space-time dimensions is even, an observer located at some distance from the source will see the light signal with clear-cut forward and backward wave fronts separated by the time interval $t-t'$. In so doing, the magnitude of the signal observed will not vary during the time provided the source works with a constant intensity. This is in good agreement with our daily experience. Another picture would be observed in odd-dimensional space-time. Of course, there will be a definite instant of time when the observer finds the source to be turned on (it is the time point when the forward wave front reaches the observer's eyes), but thereafter the source will appear to go slowly out, and no sharply definite backward front will be observed. Thus, in an odd-dimensional universe the light source, once turned on, can never be turned off. Sometimes this phenomenon is mentioned as a failure of the Huygens principle for odd dimensions.

In the context of the present work, the distinction just drawn will lead to essentially different forms for the Lorentz-Dirac equation: it will be given by a finite-order differential equation for even d's and by an integro-differential one for odd ones.

Return now to the equations of motion in the one-particle problem. The general solution to the field equation (2) is given by the sum of its particular solution and the general solution of the homogeneous equations

$$\Box A_\mu(x)=0,\quad \partial^\mu A_\mu(x)=0, \tag{6}$$

describing free electromagnetic waves incident on the particle. Using the Green's function (5) and the charge-current

[1]Strictly speaking, the parameter m entering the Lagrangian is the so called *bare mass*. The physical mass of the particle will be introduced below within the renormalization procedure.

density vector (3) we can construct the particular solution as the retarded potentials of Liénard and Wiechert:

$$A_\mu(x) = -N_d^{-1}\int G(x-y)j_\mu(y)d^n y$$

$$= -N_d^{-1}e\int G(x-x(\tau))\dot{x}_\mu d\tau. \tag{7}$$

Thus we arrive at an unambiguously determined decomposition of the electromagnetic potential into the exterior field (6) and the field created by the particle (7). The combined action of these fields on the particle is described by the Lorentz force (3), which is composed of the exterior electromagnetic field, if any, and the inevitable action of the particle upon itself, i.e., the Lorentz-Dirac force,

$$\mathcal{F}_\mu = \mathcal{F}_\mu^{ext} + \mathcal{F}_\mu^{LD}.$$

The main technical and conceptual difficulty one faces when calculating $\mathcal{F}^{LD}$ is the divergence of the integral for the retarded field (7) taken at points on the particle's trajectory—the fact is of no great surprise if one bears in mind the singularity of the Coulomb potential associated with a point charge. This problem is closely related to another one, namely, the problem of the electromagnetic mass of an electron and sometimes the two are even identified.

In the next section we extend Dirac's result to an arbitrary dimensional space-time renormalizing Green's function. We will show that in higher dimensions the elimination of infinities is not exhausted by renormalization of the mass parameter m only but brings about new renormalization constants having no analogues in the original theory (1).

III. REGULARIZATION

We start with some general remarks concerning the mathematical status of the retarded Green's function G. The Green's function (5) being a kernel of the inverse wave operator (7) is well defined when acting on a smooth compactly supported source function $j_\mu(x)$. Difficulties may arise, however, in attempting to apply the operator to a singular current like that produced by a point charge. The problem is twofold. First, except for the case of $d=3$, the Green's functions involve products of generalized functions, namely, the derivatives of a δ function multiplied by a θ function. Such products are ill defined when one treats them as generalized functions of one variable, say x^0, considering the other variables as parameters. This is the problem we will face by restricting the argument of the Green's function onto the particle's world line $x^\mu(\tau)$. The second point concerns the geometry of the domain within which the Green's function is supported. Depending on the dimension, the support coincides with, or is bounded by, the light-cone surface

$$x^2=0, \quad x_0 \geqslant 0, \tag{8}$$

which is not differentiable at $x^\mu=0$. As a result, $G(x)$ is singular at the vertex of the light cone and ill defined even in a generalized sense.

Let us illustrate this point by a simple example which, however, will play a significant role in subsequent analysis; namely, consider the generalized function $F(s)=\delta(s^2)$ defined on the real half line $s\geqslant 0$.[2] The relations $s^2=0,\ s>0$ are the one-dimensional analogues of Eqs. (8). We put by definition

$$\delta[f] = \int_0^\infty \delta(s)f(s)ds = f(0). \tag{9}$$

Then the integral associated with the linear functional F reads

$$F[f] = \int_0^\infty \delta(s^2)f(s)ds$$

$$= \frac{1}{2}\int_0^\infty \delta(s)\left(\frac{f(\sqrt{s})}{\sqrt{s}}\right)ds$$

$$= \frac{1}{2}\lim_{s\to+0}\frac{f(s)}{s}, \tag{10}$$

where $f(x)$ is a test function. In general, this integral diverges, so that the functional F is ill defined. Note, however, that the integral (10) does have meaning when evaluated on basic functions vanishing at zero. The question is how to extend the functional F, in a consistent way, from the subspace of functions vanishing at zero to the whole functional space. We will call the solution to this problem the regularization of the generalized function $F(s)$ and denote it by $\operatorname{reg} F(s)$. For example, the following expression solves the problem:

$$\operatorname{reg} F[f] = \int_0^\infty \delta(s^2)[f(s)-f(0)]ds + a_0 f(0)$$

$$= \frac{1}{2}f'(0) + a_0 f(0), \tag{11}$$

a_0 being an arbitrary constant. Indeed, the functional (11) is well defined for any basic function $f(x)$ and coincides with Eq. (10) if $f(0)=0$. In fact, Eq. (11) is the general solution to our problem since the complementary space to that of vanishing-at-zero functions (i.e., the subspace on which F comes to infinity) is one dimensional and spanned, for example, by the constant function $f(s)=1$. In regularizing the

[2]When working with generalized functions one has to fix the dual space of the basic functions, and in fact the latter is an element of the definition of the former. They are in a "dialectical" relation with each other: extending the space of the basic functions one narrows, at the same time, that of the generalized ones, and vice versa. The particular choice of a basic functional space is mainly dictated by a problem to be considered. Hereafter, one may always think of a basic function as any smooth function on the real half line. In so doing, all the derivatives at zero are understood as the derivatives on the right.

linear functional F, we just replace the infinite value $F[1]=\infty$ by any finite constant $\mathrm{reg}\,F[1]=a_0$.

This procedure can be straightforwardly extended to derivatives of the δ function. Consider the generalized function

$$F^n(s)=\delta^{(n)}(s^2)=\left(\frac{d}{ds^2}\right)^n\delta(s^2),\qquad n=1,2,\ldots. \tag{12}$$

Then we put

$$\begin{aligned}\mathrm{reg}F^n[f]=&\int_0^\infty\delta^{(n)}(s^2)\bigg(f(s)-f(0)\\&-sf'(0)-\ldots-\frac{s^{2n}}{2n!}f^{(2n)}(0)\bigg)ds+a_0f(0)\\&+a_1f'(0)+\cdots+a_{2n}f^{(2n)}(0)\\=&\int_0^\infty\delta(s)\frac{1}{2s}\left(-\frac{d}{2sds}\right)^n\bigg(f(s)\\&-\sum_{k=0}^{2n}\frac{s^k}{k!}f^{(k)}(0)\bigg)ds+\sum_{k=0}^{2n}a_kf^{(k)}(0),\end{aligned} \tag{13}$$

yields

$$\mathrm{reg}\,F^n[f]=\frac{f^{(2n+1)}(0)}{(n+1)!}+\sum_{k=0}^{2n}a_kf^{(k)}(0). \tag{14}$$

As before, the functional (12), as it stands, is defined only on functions that are $o(s^{2n+1})$ as $s\to+0$. Subtracting from a function f the first several terms of its Taylor expansion, we just project f onto the subspace of such functions. The value of the functional on the complementary $(2n+1)$-dimensional subspace is fixed by the arbitrarily chosen constants $a_0,\ldots,a_{2n}$.

The formula (14) is a particular example of a quite general mathematical procedure known as the regularization of divergent integrals or generalized functions. The procedure is applicable to a wide class of singular functionals, in particular, to functionals with polynomial singularities [12].

For our purposes it is instructive to rederive expression (14) in another way, perhaps more familiar for physicists; namely, using the sequential approach to generalized functions one may represent the δ function (9) as

$$\delta(s)=\lim_{a\to+0}\frac{e^{-s/a}}{a}. \tag{15}$$

Substituting this representation into Eq. (10) we get a one-parametric family of well-defined functionals until $a\neq0$:

$$F_a[f]=\int_0^\infty\frac{e^{-s^2/a}}{a}f(s)ds. \tag{16}$$

In so doing, $F[f]=\lim_{a\to0}F_a[f]$. For this reason we refer to a as the regularization parameter. Given the basic function $f(s)$, the integral (16) defines a meromorphic function in $\sqrt{a}$ given by the Laurent series

$$\begin{aligned}F_a[f]&=a^{-1/2}\int_0^\infty e^{-t^2}f(t\sqrt{a})dt\\&=\sum_{n=0}^\infty a^{(n-1)/2}\frac{f^{(n)}(0)}{n!}\int_0^\infty e^{-t^2}t^ndt\\&=\frac{1}{2}\sum_{n=0}^\infty a^{(n-1)/2}\frac{f^{(n)}(0)}{n!}\Gamma\left(\frac{n+1}{2}\right).\end{aligned} \tag{17}$$

In the limit $a\to0$ of switching regularization off, the singularity of the functional $F[f]$ appears as a simple pole, while the nonvanishing term of the regular part of Eq. (17) coincides with Eq. (11) if we take $a_0=0$. Similarly, the delta-shaped sequence (15) defines a holomorphic function in a with coefficients being the generalized functions in s,

$$\delta_a(s)=\frac{e^{-s/a}}{a}=\sum_{n=0}^\infty(-a)^n\delta^{(n)}(s). \tag{18}$$

Substitution of the last expression into Eq. (16) gives

$$F^n[f]=\lim_{a\to0}\frac{(-1)^n}{n!}\frac{d^nF_a[f]}{da^n}. \tag{19}$$

We see that the regularization (14) of the functionals F^n corresponds to the replacement

$$a^{-(k+1)/2}\to a_k,\quad k=0,1,\ldots,n \tag{20}$$

of all the poles in expression (19) by arbitrary finite constants.

Thus the sequential approach to the problem leads to the same finite part for the functional to be regularized and contains the same ambiguity in the final definition as the scheme based on direct subtraction.

IV. RENORMALIZATION AND THE LORENTZ-DIRAC FORCE

Now we are going to explicitly calculate the Lorentz-Dirac force $\mathcal{F}^{LD}$. As the results drastically differ in odd and even dimensions, we consider these cases separately.

A. Even-dimensional space-time

The retarded Green's functions are given by the first line in Eq. (5). We start with the case of $d=4$. The expression for the LD force follows from Eqs. (7) and (3):

$$\begin{aligned}\mathcal{F}_\mu^{LD}(s)=4e^2Dx^\nu(s)\int\theta(x^0(s)-x^0(\tau))\,\delta'(\{x(s)\\-x(\tau)\}^2)\{x(s)-x(\tau)\}_{[\nu}\dot{x}_{\mu]}(\tau)d\tau\end{aligned} \tag{21}$$

where the square brackets mean antisymmetrization. Since for the massive particle the equation $\{x(s)-x(\tau)\}^2=0$ means $x^\mu(s)=x^\mu(\tau)$, which in turn implies $s=\tau$, the integrand is supported at the point $s=\tau$. Changing the integration variable $\tau\to s-\tau$ we get a singular integral of the form

$$\mathcal{F}^{LD}_\mu(s)=4e^2Dx^\nu(s)\int_0^\infty \delta'(\{x(s)-x(s-\tau)\}^2)\times\{x(s)-x(s-\tau)\}_{[\nu}\dot{x}_{\mu]}(s-\tau)d\tau \qquad (22)$$

and the singularity comes from the derivative of the delta function. Following the general regularization prescription we replace δ' by an appropriate sequence of smooth functions. For example, from Eq. (18) it follows that

$$\delta'(s)=-\lim_{a\to+0}\frac{\partial}{\partial a}\frac{e^{-s/a}}{a},\quad s\geqslant 0. \qquad (23)$$

This leads to the regular expression for the LD force:

$$\mathcal{F}^{LD}_\mu(s,a)=-4e^2Dx^\nu(s)\frac{\partial}{\partial a}\int_0^\infty e^{-\{(x(s)-x(s-t)\}^2/a}\times\{x(s)-x(s-t)\}_{[\nu}\dot{x}_{\mu]}(s-t)\frac{dt}{a},$$

$$\mathcal{F}^{LD}_\mu(s)=\lim_{a\to+0}\mathcal{F}^{LD}_\mu(s,a). \qquad (24)$$

Evaluating this integral by the Laplace method we get an asymptotic expansion for $\mathcal{F}^{LD}_\mu(s,a)$ in half-integer powers of a. The actual calculations are considerably simplified if one notes that the form of the integral is invariant under reparametrizations. So we may assume that the proper time τ satisfies the additional normalization condition $\dot{x}^\mu\dot{x}_\mu=1$. In this gauge, many of the terms vanish in the Taylor expansion for the exponential and preexponential factors in (24), leaving us with

$$\{x(s)-x(s-\tau)\}^2=\dot{x}^2\tau^2-\frac{1}{12}(D^2x)^2\tau^4+\frac{1}{12}D^2x\cdot D^3x\,\tau^5+\left(\frac{1}{45}(D^3x)^2+\frac{1}{40}D^2x\cdot D^4x\right)\tau^6+O(\tau^7),$$

$$\begin{aligned}\{x(s)-x(s-\tau)\}_{[\nu}\dot{x}_{\mu]}(s-\tau)&=-\frac{1}{2}D^2x_{[\mu}Dx_{\nu]}\tau^2+\frac{1}{3}D^3x_{[\mu}Dx_{\nu]}\tau^3\\&-\left(\frac{1}{8}D^4x_{[\mu}Dx_{\nu]}+\frac{1}{12}D^3x_{[\mu}D^2x_{\nu]}\right)\tau^4\\&+\left(\frac{1}{30}D^5x_{[\mu}Dx_{\nu]}+\frac{1}{24}D^4x_{[\mu}D^2x_{\nu]}\right)\tau^5+O(\tau^6).\end{aligned} \qquad (25)$$

Substituting these expressions back into Eq. (24), we get an integral of the form (17). The result of integration reads

$$\mathcal{F}^{LD}_\mu(a)=e^2\dot{x}^2\sum_{k=2}^\infty(-1)^{k+1}\frac{(k-1)^2}{k!}\Gamma\left(\frac{k+1}{2}\right)\times f^{(k)}_\mu(Dx,\dots,D^kx)a^{(k-3)/2}. \qquad (26)$$

The explicit expressions for the first several terms are given by

$$\begin{aligned}f^\mu_{(2)}&=D^2x^\mu,\\ f^\mu_{(3)}&=D^3x^\mu+(D^2x)^2Dx^\mu,\\ f^\mu_{(4)}&=D^4x^\mu+\frac{3}{2}(D^2x)^2D^2x^\mu+3D^2x\cdot D^3xDx^\mu,\\ f^\mu_{(5)}&=D^5x^\mu+\frac{5}{2}(D^2x)^2f^\mu_{(3)}+\frac{15}{2}D^2x\cdot D^3xD^2x^\mu\\&\quad+[4D^2x\cdot D^4x+3(D^3x)^2]Dx^\mu,\\&\quad\cdots\\ f^\mu_{(k)}&=D^kx^\mu+\cdots,\\&\quad\cdots .\end{aligned}$$

Notice that in view of the reparametrization invariance of the model the vector of the regularized LD force is transverse to the particle's velocity, i.e., $\dot{x}^\mu\mathcal{F}_\mu(s,a)=0$. Upon removing the regularization only the first two terms of the series (26) survive—one singular and one finite—which leads to the well-known LD equation for a $d=4$ point charge interacting with its own field

$$\left(m+\frac{e^2}{4}\sqrt{\frac{\pi}{a}}\right)D^2x^\mu=\frac{2}{3}e^2[D^3x^\mu+(D^2x)^2Dx^\mu],\quad a\to 0. \qquad (27)$$

The right hand side (RHS) of this equation describes the back reaction of the particle upon radiating of electromagnetic waves and this is more than just an interpretation. Notice that our calculations, being as rigorous as possible, contain, however, an apparent ambiguity related to the definition of Dirac's δ function on the half line (9). This function(al) results from the restriction of the retarded Green's function to the particle's world line and appears to be ill defined even in the sense of generalized functions, as has already been mentioned at the beginning of Sec. III. Indeed, starting with an arbitrary delta-shaped sequence defined on the whole real line, one may restrict the corresponding functional to test functions vanishing identically at $s<0$. The different sequences will then lead to the different results, distinguished from each other by an overall constant factor. For example, any symmetric (with respect to zero) approximation for $\delta(s)$ will give the additional 1/2 multiplier in the RHS of Eq. (9). However, requiring conservation of the total energy of the system, one has to equate the work of the LD force with the energy of the electromagnetic field *radiated* by the accelerating particle, which immediately leads to our convention for the δ function on the half line. This is the energy conservation argument that is frequently used to derive the LD equation in four dimensions (see, e.g., [1,4,6]).

The physical interpretation for the infinite contribution in the LHS of Eq. (27) is also obvious: its appearance reflects the infinite energy, or mass, of the field *adjunct* to the particle. Within the paradigm of renormalization theory this singularity is removed by a simple redefinition of the particle's mass: one just replaces the sum of the unobservable bare mass m and the infinite contribution due to the interaction by a finite (experimental) value,

$$m_{expt}=m+\frac{e^2}{4}\sqrt{\frac{\pi}{a}}, \tag{28}$$

so that the renormalized action of the $d=4$ particle takes the form

$$S^{(4)}_{renorm}=\left(m_{expt}-\frac{e^2}{4}\sqrt{\frac{\pi}{a}}\right)\int d\tau\sqrt{\dot{x}^2}. \tag{29}$$

This means that classical electrodynamics in four dimensions is a multiplicatively renormalizable theory.

The next task is to try to extend this result to even higher dimensions using the explicit expressions for the retarded Green's functions (5). To do this one has no need to repeat all the calculations from the very beginning. In view of the relations (18) and (19) the desired expressions for the LD force can be derived in any even d by successive differentiation of the universal series (26) obtained in $d=4$. More precisely,

$$\mathcal{F}^{(d)}_{\mu}=\frac{2^{d-3}(-1)^{d/2}}{(d-2)!}\left(\frac{\partial}{\partial a}\right)^{(d-4)/2}\mathcal{F}^{LD}_{\mu}(a)|_{a=0}. \tag{30}$$

In the case of six dimensions the corresponding equation of motion reads

$$\begin{aligned}\left(m+a^{-3/2}\frac{e^2\sqrt{\pi}}{24}\right)D^2x^{\mu}&\\ =-\frac{4e^2}{45}\bigg(D^5x^{\mu}&+\frac{5}{2}(D^2x)^2[D^3x^{\mu}+(D^2x)^2Dx^{\mu}]\\ &+\frac{15}{2}D^2x\cdot D^3xD^2x^{\mu}+[4D^2x\cdot D^4x+3(D^3x)^2]Dx^{\mu}\bigg)\\ &+a^{-1/2}\frac{e^2 3\sqrt{\pi}}{64}\bigg(D^4x^{\mu}+\frac{3}{2}(D^2x)^2D^2x^{\mu}\\ &+3D^2x\cdot D^3xDx^{\mu}\bigg).\end{aligned} \tag{31}$$

In addition to the infinite mass term we observe a new type of divergence involving fourth-order derivatives. It is interesting to note that both the divergences are Lagrangian, i.e., they can be canceled out by adding appropriate counterterms to the initial Lagrangian (1), so that the renormalized action reads

$$\begin{aligned}S^{(6)}_{renorm}=-\left(m_{exp}-a^{-3/2}\frac{e^2\sqrt{\pi}}{24}\right)\int d\tau\sqrt{\dot{x}^2}&\\ -a^{-1/2}\frac{e^2 3\sqrt{\pi}}{128}\int d\tau\sqrt{\dot{x}^2}(D^2x)^2.&\end{aligned} \tag{32}$$

Contrary to this, the finite part of the LD force containing fifth derivatives of the x's cannot be represented as the variation of a Lorentz-invariant functional. These results generally agree with the previous analysis of [9], where explicit expressions for the 6D LD force and the counterterm in Eq. (32) were obtained from the requirements of energy conservation and reparametrization invariance. (The radiation reaction force (31) differs from that of [9] just by an overall coefficient. This seems to be because of minor inaccuracy in the normalization of 6D Liénard-Wiechert potentials accepted in [9].)

This situation is general and nothing changes this picture as the number of dimensions increases; namely, for any even d the finite part of the LD force is given by a polynomial function in the derivatives of $x^{\mu}(\tau)$ up to the $(d-1)$ order inclusive. Note that the higher (odd) derivative $D^{d-1}x^{\mu}$ enters linearly into LD force and therefore no variation principle for the LD equation exists. This reflects the dissipative character of the system losing energy due to the radiation. In addition, there are $d/2-1$ divergent terms, ''the most singular of which'' corresponds to infinite electromagnetic mass of the particle, while the interpretation for the other terms is not so simple as structures of such types are lacking in the original theory. By analogy with four and six dimensions one may expect that all the singularities are Lagrangian and can be removed by adding appropriate counterterms to the action (1). This appears to be the case. Indeed, since the Maxwell equations (2) are linear we may resolve them in the general form (7) and, substituting the result back into the action functional (1), get a functional of the particle's trajectory only:

$$S=-m\int d\tau\sqrt{\dot{x}^2}-\frac{1}{2}N_d^{-1}\int d^dx \times\int d^dy j^\mu(x)G(x-y)j_\mu(y)$$

$$=-m\int d\tau\sqrt{\dot{x}^2}-\frac{e^2}{2}N_d^{-1}\int ds \times\int d\tau\dot{x}^\mu(s)G(x(s)-x(\tau))\dot{x}_\mu(\tau). \quad (33)$$

The first term is the usual action functional of a free scalar particle and the second describes the self-action. Since the retarded Green's function is localized on the light cone we can explicitly perform (after a suitable regularization) one integration in the double integral and get the Lagrangian model for the relativistic particle with higher derivatives. It is not hard to check that in the cases of $d=4,6$ the regularized self-action term in Eq. (33) exactly reproduces the counterterms in the corresponding action functionals (29),(32). As to the LD force, it cannot be obtained from the higher derivative model since only the symmetric part of the retarded Green's function actually enters into the nonlocal action (33). It seems very likely that the same situation occurs in higher dimensions as well.

To summarize, for even dimension $d>4$ one is forced to extend the original Lagrangian of the free relativistic particle by the addition of $d/2-2$ extra (higher derivative) terms in order to get a renormalizable theory, so that the whole renormalization procedure involves, together with the physical mass, $d/2-1$ arbitrary constants.

B. Odd-dimensional space-time

The Green's function is given by a product of a θ function and a singular analytical expression (5). According to this, the gradient of the Green's function entering the strength tensor $F_{\mu\nu}$ of the adjunct electromagnetic field consists of structures proportional to θ and δ functions. This allows one to decompose the corresponding LD force into local and nonlocal parts and both of these parts are generally singular. In principle, proceeding by analogy with the even-dimensional case, one can remove these singularities with the help of appropriately chosen counterterms. However, the nonlocal character of the particle's self-action in odd dimensions gives rise to a specific question about the admissible choice for the counterterms. For the sake of simplicity, we restrict ourselves to illustrating the problem and its solution for the case of the $2+1$ particle. The complete analysis will be given elsewhere.

The local and nonlocal parts of the LD force are given by the integrals

$$\mathcal{F}^\mu_{nonlocal}=e^2Dx_\nu(s)\int_{-\infty}^s d\tau\left(\frac{[x(s)-x(\tau)]^{[\mu}\dot{x}(\tau)^{\nu]}}{|x(s)-x(\tau)|^3}\right), \quad (34)$$

$$\mathcal{F}^\mu_{local}=-2e^2Dx_\nu(s)\int_{-\infty}^s d\tau \times\left(\frac{\delta([x(s)-x(\tau)]^2)[x(s)-x(\tau)]^{[\mu}\dot{x}^{\nu]}(\tau)}{|x(s)-x(\tau)|}\right). \quad (35)$$

Near the coincidence limit $s\to\tau$ the numerator and denominator of the integrand (34) behave as $(s-\tau)^2$ and $(s-\tau)^3$, respectively, so that the integral diverges logarithmically, as one would expect for the two-dimensional Coulomb potential. Using the asymptotic equality

$$2[x(s)-x(\tau)]^{[\mu}\dot{x}^{\nu]}(\tau)\sim D^2x^{[\nu}(s)Dx^{\mu]}(s)[x(s)-x(\tau)]^2\sqrt{\dot{x}^2(\tau)}, \quad s\to\tau, \quad (36)$$

we can extract the infinity as follows:

$$\mathcal{F}^\mu_{nonlocal}=e^2Dx_\nu(s)\int_{-\infty}^s d\tau\left(\frac{[x(s)-x(\tau)]^{[\mu}\dot{x}(\tau)^{\nu]}}{|x(s)-x(\tau)|^3}-\frac{D^2x^{[\nu}(s)Dx^{\mu]}(s)\sqrt{\dot{x}^2(\tau)}}{2|x(s)-x(\tau)|}\right)+\delta m D^2x^\mu(s), \quad (37)$$

where

$$\delta m=\frac{e^2}{2}\int_{-\infty}^s\frac{d\tau\sqrt{\dot{x}^2(\tau)}}{|x(s)-x(\tau)|}=\infty. \quad (38)$$

Now the first integral is regular and describes the nonlocal self-action of the point charge, whereas the second term gives rise to the infinite mass renormalization.

In this simple case of the $2+1$ particle, the form of the counterterm is uniquely determined for reasons of reparametrization invariance, correct physical dimension, proper short-distance behavior, and the additional requirement for the renormalized LD force to depend on τ via the particle's trajectory $x(\tau)$ only.[3] Relaxing the last condition, one may find many other admissible subtraction schemes yielding the different result for the renormalized force. For example, in [10] it is proposed to cancel the divergence by the mass counterterm $\delta m D^2x^\mu$ with

[3]In fact, the latter condition can be understood as a particular form of the reparametrization invariance requirement, if one adopts the "passive" viewpoint on transformations: the proper time parameter τ is chosen only once, while the gauge transformation acts directly on the coordinate functions $x(\tau)\to x'(\tau)=x(f(\tau))$.

$$\delta m = \frac{e^2}{2}\int_{-\infty}^{s}\frac{d\tau}{|s-\tau|}, \tag{39}$$

which does not satisfy the above condition and results in the explicit dependence of the renormalized LD force on the world-line parameter. It can easily be seen that the two subtraction schemes (38) and (39) are not equivalent to each other even taking account of any possible reparametrizations.

As to the local part of LD force (35), it turns out to be finite and proportional to the free equations of motion,

$$\mathcal{F}^{\mu}_{local} = \frac{e^2}{2}D^2 x^{\mu}. \tag{40}$$

Note that here we do not worry about the "right" definition for the δ function on the half line as, in any case, this contribution is absorbed by the renormalization of the mass. It would be interesting to compare the work produced by the renormalized force (37) with the loss of energy due to the radiation.

The main lesson to be learned from this consideration is that, despite the nonlocal character of the self-action force in $2+1$ dimensions, the divergent part of the LD force is local and even Lagrangian. There is no doubt that the same conclusion remains true for any higher odd dimension.

V. CONCLUDING REMARKS

In this paper we considered the derivation of the Lorentz-Dirac equation for a point charge in various dimensions. In any even d the radiation reaction is given by the simple general formula (30) implying only the formal differentiation of the $d=4$ LD force by the regularization parameter. It has been shown that, contrary to the quantum electrodynamics in $d>4$, the classical electrodynamics of point particles is a renormalizable theory, although nonmultiplicatively, beyond $d=4$. The necessity of extra counterterms (involving higher derivatives) in addition to that responsible for the mass renormalization (attributed naturally to the infinite energy of the field surrounding a point charge) seems rather counterintuitive but this is a must for obtaining a reasonable theory.

The results of the paper can be extended at least in two directions. First, one may consider a supersymmetric or higher-spin generalization for the relativistic particle. Models with N-extended world-line supersymmetry were studied in Refs. [13,14]. It has been shown that after quantization these models can be consistently interpreted as relativistic spinning particles of spin $N/2$. By analogy with the quantum field theory, one may expect that inclusion of supersymmetry will result in correction of some singularities or will even lead to finite models. If this is the case, taking account of spin can give rise to a fully consistent theory of charged point particles. The consistent interactions of massive arbitrary-spin particles with exterior fields including higher dimensions have been constructed in [16,17]. Although spin induced radiation is known, its back reaction remains an open question even in $d=4$.

The second option is to extend the above analysis to a p-brane system universally coupled to a p-form field and other background fields. In this setup one may raise the question of classical stability for such a system. At the linear level the problem was studied in Ref. [15], where a full set of constraints on masses and couplings was established for a 0-brane minimally coupled to a multiplet of vector and scalar fields. It is anticipated that by generalizing this analysis to the case of an extended object we will get a certain restriction on the background fields in the form of local equations of motion. If so, this may shed new light upon the origin of low energy effective field equations in the string and brane world.

Note added. In the preceding paper, Gal'tsov comments on the validity of our assertion on renormalizability of classical electrodynamics in higher dimensions as it is *"based on an original regularization due to the authors for derivatives of the delta function, which is likely to be incorrect"*. From Sec. III of our paper, one may find that we use the standard regularization procedure for divergent integrals of generalized functions. See the textbook by Gelfand and Shilov [12]. What is more, Eq. (30) for the LD force (which we have derived in any even dimension by regularizing the equations of motion) is in line with the results for $d=6$ obtained earlier by a completely different method [9].

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